Linear Algebra I
Exercise Sheet 5: Inner Products

Inner Products

Question 1. For the following vectors u and v in the inner product spaces $\big(V, \langle , \rangle\big)$, find $\|u\|$, $\|v\|$, $\|u - v\|$, and $\langle u, v\rangle$:

- $V = \mathbb{R}^3$ with the standard dot product and $u = (1, 2, 3)$, $v = (7, -1, 0)$, and also $u = (2, -1, 1)$, $v = (1, 3, -2)$;
- $V = \mathbb{R}^2$ with the standard dot product $u = (1, 0)$ and $v = (0, 1)$;
- $V = \mathbb{R}^2$ with the inner product

$$\langle (u_1, u_2), (v_1, v_2)\rangle = 2u_1v_1 + 3u_2v_2,$$

  and $u = (1, 2)$, $v = (3, -1)$;
- $V = \{a_0 + a_1x + a_2x^2 \,|\, a_i \in \mathbb{R}\}$ with inner product

$$\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2\rangle = a_0b_0 + 4a_1b_1 + 18a_2b_2$$

  and $u = 1 + x^2$ and $v = x - x^2$;
- $V = \{a_0 + a_1x + a_2x^2 \,|\, a_i \in \mathbb{R}\}$ with inner product

$$\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2\rangle = a_0b_0 + 2a_1b_1 + 5a_2b_2,$$

  and $u = 1 + 2x - x^2$ and $v = 2 - x + 3x^2$.
- let V be the space of continuous functions defined on $[0, 1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx,$$

  and $u = 1$, $v = x$.

Question 2. Recall that

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}.$$

For each of the following pairs of vectors find the angle between them:

- $V = \mathbb{R}^2$ with the standard dot product and $u = (1, 1)$, $v = (1, -1)$.

- $V = \mathbb{R}^3$ with the standard dot product and $u = (1, 2, 2)$, $v = (2, 1, -2)$.
- $V = \{a_0 + a_1 x \,|\, a_i \in \mathbb{R}\}$ with

$$\langle a_0 + a_1 x, b_0 + b_1 x \rangle = a_0 b_0 + a_1 b_1,$$

and $u = 1 + x$, $v = 2 - x$.

Inner Product Spaces

Question 3. For the following, determine if $\big(V, \langle , \rangle\big)$ is an inner product space.

- Let V be the vector space of polynomials of degree less than n with real coefficients and with inner product defined for all $f = a_0 + \ldots + a_n x^n$, $g = b_0 + \ldots + b_n x^n \in V$ by:
- $V = \mathbb{R}^2$ and

$$\langle u, v \rangle = u_1 v_1 - u_2 v_2;$$

- $V = \mathbb{R}^2$ and

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_1 v_2;$$

- $V = \mathbb{R}^n$ and $w_i > 0$ for all $i = 1, \ldots, n$, and

$$\langle u, v \rangle = w_1 u_1 v_1 + \ldots + w_n u_n v_n;$$

- $V = \mathbb{R}^n$ and $w_i \in \mathbb{R}$ for all $i = 1, \ldots, n$, and

$$\langle u, v \rangle = w_1 u_1 v_1 + \ldots + w_n u_n v_n;$$

- let V be the space of continuous functions on $[0, 1]$ and

$$\langle f, g \rangle = f(0)g(0);$$

- $V = \mathbb{R}^n$ and

$$\langle u, v \rangle = \sum_{i=1}^{n} i\, u_i v_i;$$

1. $$\langle f, g \rangle = a_0 b_0 + \ldots + a_n b_n;$$
2. $$\langle f, g \rangle = f(c_0)g(c_0) + f(c_1)g(c_1) + \ldots + f(c_n)g(c_n),$$
   for distinct real numbers $c_0, \ldots, c_n$;
3. $$\langle f, g \rangle = f(c_0)g(c_0) + f(c_1)g(c_1) + \ldots + f(c_n)g(c_n),$$
   for real numbers $c_0, \ldots, c_n$;
4. $$\langle f, g \rangle = f(0)g(0) + f(1)g(1) + f(2)g(2);$$

5.
$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Question 4. Let $V = \mathbb{R}^2$. Determine the conditions on a, b, $c \in \mathbb{R}$ such that

$$\langle u, v\rangle = au_1v_1 + bu_1v_2 + bu_2v_1 + cu_2v_2,$$

defines an inner product on V.

Orthogonality

Question 5. Determine whether the following pairs of vectors are orthogonal with respect to the given inner product:

- $V = \mathbb{R}^3$ with the standard dot product and $u = (1, 2, -1)$, $v = (2, 0, 2)$;
- $V = \mathbb{R}^2$ with
$$\langle u, v\rangle = 3u_1v_1 + u_2v_2,$$
and $u = (1, 1)$, $v = (1, -3)$;
- $V = \{a_0 + a_1x + a_2x^2 \mid a_i \in \mathbb{R}\}$ with
$$\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\,dx,$$
and $u = 1$, $v = x$.

Question 6. For each of the following inner product spaces determine whether the given collection S vectors form an orthogonal set. If so determine whether they form an orthonormal set.

- $V = \mathbb{R}^3$ with the standard dot product and
$$S = \{(1, 0, 1), (1, 0, -1), (0, 1, 0)\};$$
- $V = \mathbb{R}^2$ with the standard dot product and
$$S = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\};$$
- In $V = \{a_0 + a_1x + a_2x^2 \mid a_i \in \mathbb{R}\}$ with
$$\langle f, g\rangle = \int_{-1}^1 f(x)g(x)dx$$
and
$$S = \{1, x, 3x^2 - 1\}.$$

Question 7. Find the orthogonal complement $W^\perp$ of the subspace W in the following inner product spaces:

- $V = \mathbb{R}^2$ with the standard dot product and

$$W = \text{Span}\{(1, 2)\};$$

- $V = \mathbb{R}^3$ with the standard dot product and

$$W = \text{Span}\{(1, 1, 0), (0, 1, 1)\};$$

- $V = \mathbb{R}^3$ with the standard dot product and

$$W = \text{Span}\{(1, 0, 1)\};$$

- $V = \{a_0 + a_1x + a_2x^2 \mid a_i \in \mathbb{R}\}$ with

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

and

$$W = \text{Span}\{1, x\}.$$

Question 8. Let $V = \mathbb{R}^3$ with the standard sot product and let

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

- Find a basis for W.
- Find a basis for $W^\perp$.
- Verify that $V = W \oplus W^\perp$.